ISRAEL JOURNAL OF MATHEMATICS, Vol. 31, No. 1, 1978

ON SHELAH'S COMPACTNESS OF CARDINALS†

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ABSTRACT

We deal with the compactness property of cardinals presented by Shelah, who proved a compactness theorem for singular cardinals. We improve that result in eliminating axiom I there and show a new application of that theorem together with a straightforward proof of it for the special case discussed. We discuss compactness for regular cardinals and show some independence results: one of them, a part of which is due to A. Litman, is the independence from ZFC + GCH of the gap-one two cardinal problem for singular cardinals.

§0. Preliminaries

Let us first review the basic notions and definitions of [7]. We deal with an algebra U (a structure with a set of operations), with a notion of freeness which is a set F of pairs of subalgebras. Let χ_2 be a cardinal big enough so that U and F are elements of $H(\chi_2)$ (the family of sets which are hereditarily of cardinality less than χ_2) and let $\boldsymbol{M}$ be an expansion of the model $(H(\chi_2), \in, =, F, U)$ which has Skolem functions. We denote by χ_0 the cardinality of the set of operations in U, and by χ_1, the cardinality of the set of relations and functions we added to get $\boldsymbol{M}$; we assume $\chi_0 \leqq \chi_1 \leqq \chi_2$. A, B, C, D denote subalgebras (we consider $\emptyset$ as a subalgebra too), M, N, denote elementary submodels of $\boldsymbol{M}$ which are also elements of $\boldsymbol{M}$, $N < M$ will mean $N < M$ (elementary submodel) and $N \in |M|$. We say that A/B is free (or A is free over B) if $\langle A, B\rangle \in F$ and A is free if $\langle A, \emptyset\rangle \in F$. A/B is λ-free if for every N, $\|N\| < \lambda$ and $A, B, \in |N|$ implies $A \cap N/B$ is free.

† This paper is based on the author's M.Sc. thesis written at The Hebrew University under the supervision of Prof. Shelah, to whom he expresses his deep gratitude.

Received June 12, 1977

THE THEOREM. The theorem we prove is: *If U satisfies the following axioms* II–VII *and λ is a singular cardinal, then every λ-free A, whose cardinality is λ, is free.* (We include axiom I only to keep the notation like that of [7].)

SET OF AXIOMS. *Ax I**. If A is free over B and $B \in N$, then $A \cap N$ is free over B.

Ax II. A is free over B iff $A \cup B$ is free over B; and always B is free over B.

Ax III. If A is free over B, and B is free over C, where $A \supseteq B \supseteq C$, then A is free over C.

Ax IV. If $A = \bigcup_{i<\lambda} A_i$, A_i $(i < \lambda)$ increasing and continuous, $A_0 \subseteq B$ and for $i < j < \lambda$, $A_j / A_i \cup B$ is free and λ is a regular cardinal, then A is free over B.

Ax V. Suppose $D \in M$, $C_i \in M$ $(i < \alpha)$, $B \subseteq D$, $A \subseteq D$, $D \subseteq C_0$, and C_i is increasing. If A is free over $(C_0 \cap M) \cup B$ and $C_i \cap M$ is free over $(C_0 \cap M) \cup D$ for $i < \alpha$, then A is free over $[(\bigcup_{i<\alpha} C_i) \cap M] \cup B$.

REMARK. (1) Note that not necessarily $A, B \in M$.

(2) Instead of $D \subseteq C_0$ we can require $M \cap (D - C_0) \subseteq B$ (just use $C_i \cup D$ instead of C_i). We shall use this version freely.

Ax VI. If A is free over $B \cup C$, and $\{A, B, C\} \subseteq N$, then $A \cap N$ is free over $(B \cap N) \cup C$.

Ax VII. If A is free over B, and $\{A, B\} \subseteq N$, then A is free over $(A \cap N) \cup B$.

The basic definition in the proof is

DEFINITION 1. We define when A is $P_\alpha(\lambda)$-free over B (or A/B is $P_\alpha(\lambda)$-free), where α is an ordinal, λ usually a regular cardinal, but sometimes a limit ordinal. We define by induction on α:

(1) $\alpha = 0$. Any pair A/B is $P_0(\lambda)$-free.

(2) $\alpha = \delta$ a limit ordinal. The pair A/B is $P_\delta(\lambda)$-free iff for every $\beta < \alpha$, A/B is $P_\beta(\lambda)$-free.

(3) $\alpha = \beta + 1$. The pair A/B is $P_\alpha(\lambda)$-free if it has a $P_\beta(\lambda)$-decomposition. A $P_\beta(\lambda)$-decomposition of A/B is a sequence A_i $(i < \delta)$ such that:

(i) A_i is increasing and continuous, $\bigcup_{i<\delta} A_i \subseteq A$ and $\operatorname{cof} \delta = \operatorname{cf} \lambda$, $A_0 \subseteq B$.

(ii) For $i \geqq j > \delta$, A_{j+1} is $P_\beta(\lambda)$-free over $A_i \cup B$.

(iii) A is free over $\bigcup_{i<\delta} A_i \cup B$.

(iv) A_{i+1} is free over B (for $i < \delta$).

(v) For $i < j < \delta$, A_{j+1} is free over $A_{i+1} \cup B$.

REMARK. (1) Notice that the definition depends on U and F only (and not on M).

(2) Note then when $\mathrm{cof}(\delta) = \mathrm{cof}(\gamma)$, A/B is $P_\alpha(\delta)$-free iff it is $P_\alpha(\gamma)$-free.

(3) Notice that if we remove the restriction in (v) that $A/A_j \cup B$ is free only for successor j_i, we get that $P_\alpha(\lambda)$-freeness (for $\alpha > 0$) implies freeness (A_s). Claim (0.1) below implies $\bigcup_{i<\delta} A_i/B$ is free and Axiom III with condition (iii) here implies A/B is free. So if we know A/B is $P_\alpha(\lambda)$-free and we want to assure it is free, the only places we lack information is over limit A_i's, where all we have is $P_\beta(\lambda)$-freeness, therefore the bigger α is, the stronger the notion of $P_\alpha(\lambda)$-freeness becomes.

(4) In fact, we use in the proof only $P_1(\lambda)$ and $P_\omega(\lambda)$-freeness.

The following technical lemmas from [7] will be used freely in the course of this proof:

Claim 0.1. *If A_i $(i < \alpha)$ is increasing and continuous, A_0 is free over B, and A_{i+1} is free over $A_i \cup B$ (for $i < \alpha$), then $\bigcup_{i<\alpha} A_i$ is free over B.*

Claim 0.2. *If $|A| = \lambda$, λ is a regular cardinal bigger than χ_1, $A = \bigcup_{i<\lambda} A_i \langle A_i : i < \lambda \rangle$ increasing and continuous for every i $|A_i| < \lambda$, then A/B is free iff there is a closed and unbounded set $S \subseteq \lambda$ such that for any $i, j \in S$, $i < j$, A_i/B and $A_j/A_i \cup B$ are free.*

Lemma 0.3. *If $A/B \cup C$ is $P_\alpha(\lambda)$-free (λ is a regular cardinal) and $\{A, B, C, \lambda, \alpha\} \cup \alpha \subseteq N$ and $\lambda \cap N$ is an initial segment of λ, then $A \cap N/B(B \cap N) \cup C$ is $P_\alpha(\delta^*)$-free where $\delta^* = \lambda \cap N =$ the order type of $\lambda \cap N =$ the first ordinal not in N.*

Lemma 0.4. *Suppose $D \in M$, $C_i \in M$ $(i < \gamma)$ and $B \subseteq D$, $A \subseteq D$, $D \subseteq C_0$, and C_i is increasing, λ a regular cardinal.*

If A is $P_\alpha(\lambda)$-free over $(C_0 \cap M) \cup B$ and $C_i \cap M$ is free over $(C_0 \cap M) \cup D$, then A is $P_\alpha(\lambda)$-free over $(\bigcup_{i<\lambda} C_i \cap M) \cup B$.

Lemma 0.5. *Suppose λ is regular, $\lambda > \chi_1$, and $\alpha \cup \lambda \cup \{A, B, \alpha, \lambda\} \subseteq N$. If A is $P_\alpha(\lambda)$-free over B, then A is $P_\alpha(\lambda)$-free over $(A \cap N) \cup B$.*

(The last three lemmas say that the notion of $P_\alpha(\lambda)$-freeness satisfies variants of Axioms VI, V, VII respectively.)

§1. The proof

The proof of the theorem in [7] is divided into two steps; the first is lemma 1.8 there, and says that under the assumptions of the theorem ($|A| = \lambda$, λ singular and A/B is λ-free) A/B is $P_\beta(\mathrm{cof}\,\lambda)$-free for every $\beta \leqq \lambda$; the second step (lemma 1.10 there) shows that every $P_\omega(\lambda)$-free pair is free (this is the case when $\chi_1 \leqq \aleph_0$, otherwise we need more assumptions on U, see [7] for details). The proof of lemma 1.10 there does not involve Axiom I, so in order to eliminate it we have to prove only the first step.

LEMMA 1.1. *Let $\langle N_i : i < \mu \rangle$ be an increasing (in the order $<$) and continuous sequence such that $\{A, B, \mu\} \cup \mu \subseteq N_1$. If $\langle A \cap N_i : i < \mu \rangle$ is a $P_0(\mu)$-decomposition of A/B then A/B is $P_\alpha(\mu)$-free for every α such that $\alpha + 1 \subseteq N_i$ for every i.*

PROOF. By induction on α. Clearly it is enough to handle $\alpha = \beta + 1$. Let us assume A/B is $P_\beta(\mu)$-free and we show $\langle A \cap N_i : i < \mu \rangle$ is a $P_\beta(\mu)$-decomposition of A/B. The demands (i), (iii), (iv), (v) in the definition of such a decomposition do not depend on β, so they hold by the assumption that it is a $P_0(\mu)$-decomposition. A/B is $P_\beta(\mu)$-free, so by Lemma 0.5 (with N_i replacing N) $A/(A \cap N_i) \cup B$ is also $P_\beta(\mu)$-free. Now we use Lemma 0.3 as A, $N_i \in N_{j+1}$, $A \cap N_i \in N_{j+1}$, and we get that $A \cap N_{j+1}/(A \cap N_i) \cup B$ is $P_\beta(\mu)$-free, so also have (ii). If $\alpha, \mu \not\subseteq N_0$ we get the $P_\alpha(\mu)$-freeness of $A \cap N_{j+1}/(A \cap N_0) \cup B$ from the conditions (i) and (iv) is the given $P_0(\mu)$-decomposition.

SCHEMA OF THE PROOF. Having this lemma we will complete the proof after we find a $P_0(\mathrm{cof}\,\lambda)$-decomposition of the above type. In order to get such a decomposition, given A, B, we define some filters on $S_\kappa(A) = \{a : a \subseteq A, |a| = \kappa\}$. Using these filters we will define for $C \in S_\kappa(A)$ "the degree of C"; the needed decomposition will be built of algebras with degree ∞. The definition of the degree will assure such algebras are free over B (so we handle condition (iv)), Lemma 1.6 will show that for such C's there are many (relative to some filter) D's such that $D/B \cup C$ is free, and using them to get the next C we handle condition (v). Lemma 1.5 shows we can get many C's with degree ∞ and $C = A \cap N$ for some N, as needed in Lemma 1.1, and in Lemma 1.7 we finally build the desired decomposition using the assumptions of the theorem.

Let A, B be fixed, $|A| = \lambda$; let μ denote regular cardinals and $\chi_1 \leqq \kappa < \lambda$.

DEFINITION 2. For any set a and $\kappa \leqq |a|$, let $S_\kappa(a) = \{b \subseteq a, |b| = \kappa\}$.

DEFINITION 3. (A) An expansion M^* of M is called a κ-expansion if it is an expansion by $\leqq \kappa$ relations and functions, and A, B, i $(i \leqq \kappa)$ are individual constants of M^*.

(B) N_i $(i < \alpha)$ is an M^*-sequence if it is increasing (by $<$) and continuous and for every $i < \alpha$, $\langle N_j : j \leqq i \rangle \in N_{i+1}$ and $N_i < M^*$.

(C) For any $\kappa < \lambda$ (and $\mu \leqq \kappa$) let $\boldsymbol{E}_\kappa(A)$ be the filter generated by sets $S \subseteq S_\kappa(A)$ called its generators such that, for some κ-expansion $\boldsymbol{M}^*$ of $\boldsymbol{M}$,

$$S = \boldsymbol{S}_\kappa(\boldsymbol{M}^*)$$

where

$$\boldsymbol{S}_\kappa(\boldsymbol{M}^*) = \left\{ A \cup \bigcup_{i<\alpha} N_i : N_i\,(i < \mu) \text{ is an } \boldsymbol{M}^*\text{-sequence}, \|N_i\| = \kappa,\ \mu < \kappa^+ \right\}$$

(so every expansion defines a generator of the filter).

(D) $E_\kappa(A)$ is generated similarly, except that the length of the M sequences can be any ordinal $< \kappa^+$; we denote its generators $S_\kappa(\boldsymbol{M}^*)$.

(E) $\boldsymbol{E}^\mu_\kappa(A)$ is generated similarly, except that we allow only $\boldsymbol{M}^*$-sequences of a fixed length—μ.

(F) $E^\mu(A)$ is generated like $\boldsymbol{E}^\mu_\kappa(A)$, but here the length of the sequence can be any $\alpha < \kappa^+$ such that $\operatorname{cof} \alpha = \mu$.

Lemma 1.2. *All the filters defined above are κ^+-complete (i.e. the intersection of less than κ^+ elements of the filter is in the filter).*

Proof. It is enough to show it for intersections of generators. Every generator is defined by a κ-expansion of $\boldsymbol{M}$. Given less than κ^+ such expansions we can define a common expansion of all of them that will still be a κ-expansion. The generator defined by such a common expansion will be included in the intersection, as every elementary submodel of this expansion will be an elementary submodel of every model in the given set of expansions, so the intersection belongs to the filter.

Lemma 1.3. *For regular κ (recall κ stands for cardinals satisfying $\chi_1 \leqq \kappa < \lambda$) the filter $\boldsymbol{E}^\kappa_\kappa(A)$ is generated also by the sets of the form*

$$\boldsymbol{S}^*_\kappa(\boldsymbol{M}) = \{A \cap \bigcup_{i<\kappa} N_i : \langle N_i : i < \kappa\rangle \textit{ is an } \boldsymbol{M}^* \textit{ sequence and } \|N_i\| < \kappa\},$$

where $\boldsymbol{M}^*$ is a χ_1-expansion of $\boldsymbol{M}$.

Proof. This is lemma 3.3 in [7] so we omit the proof here.

Definition. (A) The pair A/B is E-free (E, or $E(A)$, is a filter over a family of subsets of A) if $\{C : C \in \bigcup E, C/B \text{ is free}\} \in E$.

(B) We can replace "free" by any other property.

Remark. Obvious monotonicity results hold.

DEFINITION 5. (A) For every $\mu \leqq \kappa < \lambda$, $C \in S_\kappa(A)$, $|A| = \lambda$, and B, and filter E over $S_\kappa(A)$, we define the rank $R(C, E)$ as an ordinal or ∞, so that

(1) $R(C, E) \geqq \alpha + 1$ iff C/B is free and $\{D \in S_\kappa(A): C \subseteq D, D/C \cup B$ is free and $R(D, E) \geqq \alpha\} \neq \emptyset \bmod E$.

(2) $R(C, E) \geqq \delta$ ($\delta = 0$ or δ limit) iff C/B is free and $\alpha < \delta$ implies $R(C, E) \geqq \alpha$ (more exactly, we should write $R(C, E; A/B)$).

(B) $R(A/B, E) = \sup\{R(C, E); C \in S_\kappa(A)\}$.

(C) $R^\mu_\kappa(C) = R(C, E^\mu_\kappa)$ and $R^\mu_\kappa = R^\mu_\kappa(A/B) = R(A/B, E^\mu_\kappa)$; $\mathbf{R}^\mu_\kappa$, R^μ_κ are defined similarly.

REMARK 1.4. For such a filter E and $C \in S_\kappa(A)$, $R(C, E) = \infty$ implies

$$\{D: D \in S_\kappa(A), C \subseteq D, D/C \cup B \text{ free}, R(D, E) = \infty\} \neq \emptyset \bmod E,$$

for as $S_\kappa(A)$ is a set, the range of the degree function on its elements is bounded for such a bound α_0 and $C \in S_\kappa(A)$, $R(C, E) = \infty$ iff $R(C, E) \geqq \alpha_0$, so the remark follows from the definition.

LEMMA 1.5. *Let $\kappa^+ < \lambda$, if A/B is not $\mathbf{E}^{\kappa^+}_{\kappa^+}$-nonfree (i.e. $\{C: C \in S_\kappa(A), C/B$ is not free$\} \notin \mathbf{E}^{\kappa^+}_{\kappa^+}$) then for every $\mu \leqq \kappa$ and every κ-expansion M^*, for every $S_1 \in \mathbf{E}^{\kappa^+}_{\kappa^+}(A)$ and every $S_2 \in E^\mu_\kappa(A)$ there is $C \in S_2$ so that $R^\mu_\kappa(C) = \infty$ and C is of the form $C = D \cap N$ for D satisfying: $D \in S_1$, $D \in N$, D/B is free, and $N < M^*$, $\|N\| = \kappa$.*

(Intuitively the lemma claims that if A has many subalgebras of cardinality κ^+ free over B then it has many subalgebras of cardinality κ and degree ∞.)

PROOF. To each $C \in S(A)$ we attach a κ-expansion M^*_c, as follows: If $R^\mu_\kappa(c) = \infty$ or C_B is not free, we choose an arbitrary M^*_c, otherwise C_B is free and yet its degree has not reached ∞ so there is a reason that stopped it from climbing; that means there is a set belonging to E^μ_κ and for every element of this set, D, which contains C, $D/C \cup B$ is not free. This set includes a generator of E^μ_κ and we choose as M^*_c the κ-expansion that defines this generator.

Let M^2 be the expansion which defines the generator of E^μ_κ included in S_2, and let M^+ be a κ-expansion expanding both M^* and M^2 having the relations $P_1 = \{(c, N): C \in S_\kappa(A),\ N < M^*_c\}$ and $P_2 = \{N: N < M^2\}$; let S be $\{D \in S_{\kappa^+}(A): D/B$ is free$\}$, $S \neq \emptyset \bmod \mathbf{E}^{\kappa^+}_{\kappa^+}$ so its intersection with every element of this filter is not empty, $S_1 \in \mathbf{E}^{\kappa^+}_{\kappa^+}$ and by Lemma 1.3 $\mathbf{S}^*_{\kappa^+}(M^+)$ is also in this filter, so there is $D \in S \cap S_1 \cap \mathbf{S}^*_{\kappa^+}(M^+)$. Such a D has the form $D = A \cap \bigcup_{i<\kappa^+} N_i$ for $\langle N_i: i < \kappa^+\rangle$ an M^+-sequence such that $\|N_i\| \leqq \kappa$ (by restricting ourselves to a tail of this sequence); we can assume that for every i, $\|N_i\| = \kappa$ and $\kappa \subseteq |N_i|$. We denote $A^*_i = D \cap N_i$.

For each $C \in S_\kappa(A) \cap |N_i|$ and for each $j < i$ the sentence "There exists a model N_i^j such that $N_i^j < M_c^*$, $|N_j| \subseteq |N_i^j|$ and $\|N_i^j\| = \kappa$" is of first order in the language of M^+, it is true in M^+, and as $N_{j+1} < M^+$ and $N_j \in |N_{j+1}|$ it is true also in N_{j+1}. Such N_i^j will be even included in N_{j+1}, because N_{j+1} is an elementray submodel of a κ-expansion and $\|N_i^j\| \leqq \kappa$ (in M there are Skolem functions, so there is a function attaching to each element a one-one function from its cardinality onto it; as $\kappa \subseteq N$, N includes also the ranges of those cardinality functions with domain κ). Hence for each limit δ and each $C \in S_\kappa(A) \cap N_\delta$ we get $N_\delta < M_c^*$ and so each initial segment of $\langle N_j : i < j < \kappa^+, j \text{ limit}\rangle$ is an M_c^* sequence (for $C \in N_i$) and for $\delta < i$, δ a limit of limit ordinals, $\operatorname{cof}\delta = \mu$, $A_\delta^* \in S_\kappa^\mu(M_c^*)$. D/B is free, so by Lemma 1.2 there exists $S \subset \kappa^+$ closed and unbounded in κ^+ such that for $i, j \in S$, $i \subseteq j$, $A_j^*/A_i^* \cap B$ and A_j^*/B are free. Let W be the stationary set of elements of S which are limit of limits and has cofinality μ. $i \in W$ implies $\mathrm{R}_\kappa^\mu(A_j^*) = \infty$, because otherwise taking an increasing sequence in $\langle i_n : n < \omega\rangle$, $i_0 = i$, $i_n \in W$ for every n, we get $A_{i_{n+1}}^* \in S_\kappa^\mu(M_{A_{i_n}}^*)$ for every n, and, as $A_{i_{n+1/B}}^*$ is free, $\mathrm{R}_\kappa^\mu(A_{n+1}^*) < \mathrm{R}_\kappa^\mu(A_n^*)$ so $\langle \mathrm{R}_\kappa^\mu(A_n^*) : n < \omega\rangle$ is a decreasing sequence of ordinals, a contradiction. As every elementary submodel of M^+ is also an elementary submodel of M^2, M^+-sequences are also M^2-sequences, so for δ with cofinality μ, $A_\delta^* \in S_2$, and if we take N_i such that $D \in N_i$ and $i \in W$, then N_j, D, $N_i \cap D$ satisfies the demands of the lemma standing for N, D and C respectively.

Lemma 1.6. *For regular $\kappa > \chi_1$ if A/B is not E_κ^κ-nonfree, then for every regular $\bar\kappa$ smaller than κ and for every C such that $\mathrm{R}_{\bar\kappa}^{\bar\kappa}(C) = \infty$, $\{D \in S_\kappa(A)$: (if D/B is free then $D/B \cup C$ is free)$\} \in E_\kappa^\kappa$. That means that if $C \in S_{\bar\kappa}(A)$ and $\mathrm{R}_{\bar\kappa}^{\bar\kappa}(C) = \infty$ then there are many D's in $S_k(A)$ which are free over $B \cup C$.*

Proof. First note that for $C \in S_{\bar\kappa}(A)$, $\{D \in S_{\bar\kappa}(A) : C \subseteq D\}$ belongs to each of the filters we have defined on $S_{\bar\kappa}(A)$ because it always contains the generator defined by a $\bar\kappa$-expansion which has the elements of C as individual constants. Therefore (by Remark 1.4), given C_0, $C_1 \in S_{\bar\kappa}(A)$ satisfying $\mathrm{R}_{\bar\kappa}^{\bar\kappa}(C_0) = \infty$ and $C_0 \subsetneq C_1$, there is $D \in S_{\bar\kappa}(A)$ such that $C_1 \subseteq D$, $\mathrm{R}_{\bar\kappa}^{\bar\kappa}(D) = \infty$ and $D/B \cup C_0$ is free. Let g be a function choosing such a D for every pair (C_0, C_1). Let M^+ be a κ-expansion having g and Skolem functions for M and $P = \{N : N < M\}$ and C and a function attaching to every set x its closer to an elementary submodel for M. We will show that for every $D \in S(M^+)$ the freeness of D/B implies the freeness of $D/B \cup C$. Every $D \in S_\kappa^\kappa(M^+)$ has the form $D = A \cap \bigcup_{i<\alpha} N_i$, where $\langle N_i : i < \alpha\rangle$ is an M^+-sequence, $\operatorname{cof}\alpha = \kappa$ for each i. By Claim 1.2, if D/B is free there is $S \subseteq \kappa$ closed and unbounded there such that $i < j \in S \Rightarrow$

$A \cap N_j/(A \cap N_i) \cup B$ is free and so is $A \cap N_i/B$. We define by induction on $n < \omega$ sequences $\langle M_n : n < \omega \rangle$, $\langle i_n : n < \omega \rangle$ and $\langle C_n : n < \omega \rangle$, each C_n is in $S_{\bar\kappa}(A)$ and has degree ∞ (by $E_{\bar\kappa}^{\bar\kappa}$), the M_n's are models of cardinality $\bar\kappa$, $M_n < M_{n+1} < M$ and the $\langle i_n : n < \omega \rangle$ is an increasing sequence of ordinals in S. Let C_0 be the given C. M_n is the closer of $M_{n-1} \cup C_n \cup \{i_{n-1}, C_{n-1}, N_{i_{n-1}}, M_{n-1}\}$ to an elementary submodel of M, C_{n+1} is $g(C_n, |M_n| \cap A)$ and i_n is an ordinal high enough in S to get $N_{i_n} \supseteqq M_n$ (there is such i_n as $\|M_n\| = \bar\kappa$ and $\langle N_i : i < \alpha \rangle$ is an increasing sequence of length of cofinality $\kappa > \bar\kappa$).

Let i_ω be $\bigcup_{n<\omega} i_n$, $C_\omega = \bigcup_{n<\omega} C_n$ and $M_\omega = \bigcup_{n<\omega} M_n$. It follows that $N_{i_\omega} \supseteq C_\omega = |M_\omega| \cap A$.

To show $D/B \cup C$ is free we will use Axiom III and show it in stages:

(1) $D/(A \cap N_{i_\omega}) \cup B$ is free, because $\langle A \cap N_j : j \in S, j < i_\omega \rangle$ is an increasing and continuous (as S is closed) sequence, its union is D (as S is unbounded) and its elements are free over $(A \cap N_{i_\omega}) \cup B$ and over its union with their predecessors, so we finish by Claim 1.1.

(2) $C/C \cup B$ is free. For each i, C_i/B is free $(R_{\bar\kappa}^{\bar\kappa}(C_i) = \infty)$ $C_i/C \cup B$, so by Ax II, $C_i \cup B/C \cup B = C_i \cup C \cup B/C \cup B$ is free as well as $C_{n+1}/C_n \cup (C \cup B) C_{n+1}/C_n \cup B$ (for every n), so by Claim 1.1 we finish.

(3) $A \cap N_{i_0}/(C_\omega \cap N_{i_0}) \cup B$ is free.

Proof. $A \cap N_{i_0}/B$ is free as $i_0 \in S$. $A \cap N_{i_0} \in M_\omega$ so by Ax VIII $A \cap N_{i_0}/(A \cap N_{i_0} \cap M_\omega) \cup B$ is free, which is what we wanted. Let us from now to the end of this proof denote $A \cap N_{i_n}$ by A_{i_n}.

(4) $A_{i_0}/C_\omega \cup B$ is free. Here we will use Ax V. We will put A_{i_n}'s instead of the C_n's in the axiom, A_{i_0} instead of the D there, M_ω instead of M, A_{i_0} instead of A, and B here will take the place of B in the axiom. Let us check the conditions of the axiom, $A/(C_0 \cap M) \cup B$ becomes $A_{i_0}/(C_\omega \cap N_{i_0}) \cup B$ and we saw in (3) that it is free. $C_i \cap M/(C_0 \cap M) \cup D$ becomes $A_{i_n} \cap M_\omega/A_{i_0}$ and as A_{i_n}/A_{i_0} is free for every n and A_{i_n}, $A_{i_0} \in M_\omega$ we get the desired freeness using Axiom VI (with $B = \emptyset$, $C = A_{i_0}$, $A = A_{i_0}$, $N = M_\omega$). The conclusion of Ax V will give us the desired freeness.

(5) For every n, $A_{i_{n+1}}/(A_{i_n} \cup C_\omega) \cup B$ is free. The proof here is like the proof of (4). We start with the freeness of $A_{i_{n+1}}/A_{i_n} \cup B$ which is assured by the demand that every i_n belongs to S. We then use Ax VII (with M_ω as N there) and get $A_{i_{n+1}}/A_{i_n} \cup (A_{i_{n+1}} \cap M_\omega) = A_{i_{n+1}}/A_{i_n} \cup (C_\omega \cap N_{i_{n+1}}) \cup B$ is free and then use Ax V just like we did in (4).

(6) $A_{i_\omega}/C \cup B$ is free. This is an immediate consequence of (4) and (5) using Claim 1.1.

Collecting all these results we have the freeness of $D/A_{i_\omega} \cup B$, $A_{i_\omega} \cup B/C_\omega \cup B$, and $C_\omega \cup B/C \cup B$ so by Ax III, $D/C \cup B$ is free.

LEMMA 1.7. *If $|A| = \lambda$ is singular, and $\{\kappa : A/B$ is not $E_\kappa^{\kappa^+}$-nonfree$\}$ is unbounded in λ, and $\alpha < \lambda$, then A/B has a $P_0(\operatorname{cof}\lambda)$-decomposition of the form $\langle A \cap N_i : i < \operatorname{cof}\lambda\rangle$ where $\langle N_i : i < \operatorname{cof}\lambda\rangle$ is increasing (in the order $<$) and continuous and for every $i < 0$, $\alpha \cup \{A, B, \operatorname{cof}\lambda\rangle \cup \operatorname{cof}\lambda \leqq N_i$. (Note that if A/B is λ-free then for every $\kappa < \lambda$ A/B is not $E_\kappa^{\kappa^+}$-nonfree.)*

PROOF. We build $\langle N_i : i < \operatorname{cof}\lambda\rangle$ by induction on i. Let $A = \{a_\alpha : \alpha < \lambda\}$, and $\langle \kappa_i : i < \operatorname{cof}\lambda\rangle$ is an increasing sequence of cardinals bigger than χ_1, its limit is λ, and for every i A/B is not $E_{\kappa_i}^{\kappa_i^+}$-nonfree. At each step $i+1$ we want that $|A \cap N_{i+1}| = \kappa_{i+1}$ and $R_{\kappa_{i+1}}^{\kappa_{i+1}}(A \cap N_{i+1}) = \infty$. For $i = 0$ let $N_0 = \emptyset$, and for i limit $N_i = \bigcup_{j<i} N_i$. For $i = j+1$ let C be $(N_j \cap A) \cup \{a_\alpha : \alpha < \kappa_i\}$ and S_0 will be $\{D \in S_{\kappa_i^+}(A)$: $\alpha < j \Rightarrow (D/B$ free $\Rightarrow D/B \cup (A \cap N_{\alpha+1})$ is free)$\}$, as the degree of every $A \cap N_{\alpha+1}$ $(\alpha < j)$ is ∞ (relative to the suitable filter of course) and A is not $E_\kappa^{\kappa^+}$-nonfree. We can use Lemma 1.6 and the κ_i^{++}-completeness of $E_{\kappa_i}^{\kappa_i+}$ to get $S_0 \in E_\kappa^{\kappa^+}$.

Let M^* of an expansion of M be $\langle N_\alpha : \alpha < i\rangle$ and the ordinals smaller than κ_i (it is a κ_i-expansion); $S_1 = S_0 \cap \{A \cap N : N < M^*, \|N\| = \kappa_i^+\}$ belongs to $E_\kappa^{\kappa^+}$ (as an intersection of two elements of the filter).

Let $S_2 = \{G : E \in S_{\kappa_i}(A), G \supseteq C\}$, S_2 belongs to $E_{\kappa_i}^{\kappa_1}$. Now using Lemma 1.5 (κ_i stands for κ there) we get N and D such that $R_{\kappa_i}^{\kappa_i}(D \cap N) = \infty$, $D \in N$, $N < M^*$, D/B is free, $\|N\| = \kappa_i$, $D \in S_1$, and $D \cap N \in S_2$ so $D \cap N = A \cap N'$ for $N' < M^*$. Let $N_i = N' \cap N$; it follows that $N_i < M^*$, $R_{\kappa_i}^{\kappa_i}(A \cap N_i) = \infty$ so $A \cap N_i/B$ is free, $D \in S_0$ is free over B and $D/(A \cap N_{\alpha+1}) \cup B$ is free (as $D \in S_0$). Clearly $D/B \cup (A \cap N_{\alpha+1})$ is free for every $\alpha < j$. Also every $N_{\alpha+1}$ is an element of $N \cap N'$ (it is an individual constant), so we can use Ax VI to get the freeness of $A \cap N_j/B \cup (A \cap N_{\alpha+1})$. Now we choose j such that $\operatorname{cof}\lambda \cup \alpha < \kappa_j$ and the desired sequence of models will be $\langle N_i' : i < \operatorname{cof}\lambda\rangle$, where $N_0' = N_0$, $N_i' = N_{i+j}$.

§2. An application: order types which are not a union of $\aleph_0$ well orderings

We give here one more application of the compactness theorem (there are many applications in [7]). This is an answer to a question of Baumgartner [2], and for the theorem used here we show a much simpler proof than the proof of the compactness theorem.

In [2] Baumgartner deals with order types that cannot be represented as a union of $\aleph_0$ well orderings. He presents a subclass of those orders: The class of all such order types with the property that every uncountable subtype of φ contains a copy of ω_1. He denotes this subclass Φ_4 and asks (problem 1 there) if there exist types in Φ_4 whose cardinality is a singular cardinal but every subtype of smaller cardinality can be represented as a countable union of well orderings.

We give a negative answer by showing that the existence of such a type is equivalent to a certain incompactness property (in its cardinality) and that the compactness theorem holds for this property.

Let N be the class of all continuous functions from successor countable cardinals to the ordinals ($f \in N \to \operatorname{Dom} f = \alpha + 1$, α countable). An order $<$ is defined on N; $g < f$ iff $f \subsetneqq g$ or for the first β such that $f(\beta) \neq g(\beta)$, $g(\beta) < f(\beta)$. Given $S \subseteq N$, $T(S)$ is the tree generated by S ($T(S) = \{f \in N : (\exists g \in S)(f \subseteq g)\}$ with the inclusion as the order). Theorem 2.1 in [2] states that every order type φ without a decreasing sequence of length ω_1 can be represented as $S \subseteq N$ (with the order $<$) such that in $T(S)$ there is no path of length ω_1. (In particular this holds for types which can be represented as a union of $\aleph_0$ well orderings and for types in Φ_4.) The proof is by well orderings φ, and by defining by induction $f : \varphi \to N$, an embedding.

Notation. A branch in a tree is the set of predecessors of a limit node.

Theorem 2.1. *For $S \subseteq N$ such that in $T(S)$ there is no path of length ω_1, the order type of $(S, <)$ can be represented as a union of $\aleph_0$ well orderings iff there is a function that attaches to every branch of $T(S)$ a final segment of it, such that for incomparable branches the segments are disjoined.*

Proof. We prove the if part by showing that for some μ, $\operatorname{cof}(\mu) \geqq \lambda = |T(S)|$, $(S, <)$ can be embedded in $({}^{\omega>}\mu, <)$ (that is the set of finite sequences of ordinals in μ with the order defined on N). This is enough as ${}^{\omega>}\mu = \bigcup_{n<\omega} {}^n\mu$ and $<$ well orders ${}^n\mu$ (the sequence of length n is μ) for every n. The proof is by induction of $\lambda = |T(S)|$. For $\lambda \leqq \aleph_0$ $(S, <)$ is countable and as $({}^{\omega>}\mu, <)$ is dense it can be embedded there. Assuming our claim holds for cardinals smaller than λ we represent $T(S)$ as a union of an increasing and continuous sequence of trees of smaller cardinality each of which contains all initial segments of its branches $T(S) = \bigcup_{i<\lambda} T_i$. When a tree with a function F choosing disjoint final segments for its branches is presented as $T = \bigcup T_i$ (increasing continuous and $|T_i| < |T|$ for every i), there is a closed and unbounded set C in $|T|$ such that for $\alpha \in C$ and $\beta > \alpha$, T_β has a function F^β choosing disjoint final segments of its branches which are all disjoint to every branch in T_α. (This is not hard to see but just the same will be proved later (3.6).) Thus we represent $T(S)$ as $T(S) = \bigcup_{i \in C} T_i$ for such C.

By induction on i we build functions $f_i : T_i \to {}^{\omega>}\mu$ such that $j < i$ implies $f_j \subseteq f_i$ and each f_i is order preserving; $\bigcup_{i<\lambda} f_i$ will be the desired embedding of $T(S)$.

We assume f_i is defined and define f_{i+1}. By induction on the height of the vertices in T_{i+1} let us show that for every one of them there is a first element in the cut it defines in $(T_i, <)$: If $t \in T_{i+1}$ is a limit, $F^{i+1}(t)$ is defined, its first element belongs to T_{i+1} (as the T_j's contain initial segments of their branches), does not belong to T_i (this is the demand on the F^β's) and thus defines the same cut as t defines, and as the first element of $F^{i+1}(t)$ has smaller height than t the induction hypothesis assures the cut has a first element. For t of height $\alpha > \beta s\hat{}\langle\alpha\rangle \not\in T_i$ (where the height of s is α); if $s \not\in T_i$ then s and t define the same cut, so the induction hypothesis takes care. So assume $s \in T_i$, if for every $\alpha > \beta s^{\vee}\langle\alpha\rangle \not\in T_i$ then s is the first element in the cut it defines, otherwise let γ be the first ordinal such that $s\hat{}\langle\gamma\rangle \in T_i$, then $s\hat{}\langle\gamma\rangle$ is the first element in the cut. In order to define f_{i+1} we show that given a set of elements of T_{i+1} which all define the same cut in T_i, we can embed them (in an order preserving way) in ${}^{\omega>}\mu$ such that their images will define the same cut relative to $f_i(T_i)$. Let a be the first element in T_i above the cut. As $|T_i| < \lambda$ and $\operatorname{cof}(\mu) > \lambda$ there exists a first $\alpha < \lambda$ such that $\beta \geqq \alpha$ implies $f_i(a)\hat{}\langle\beta\rangle \not\in f_i(T_i)$, the tree of vertices bigger (in ${}^{\omega>}\mu$) than $f_i(a)\hat{}\langle\alpha\rangle$ is isomorphic to ${}^{\omega>}\mu$, and as the cardinality of the set we want to embed is smaller than λ the induction hypothesis assures us we can imbed it there (clearly all the vertices in this tree define the same cut in $F_i(T_i)$, the cut whose first element is $f_i(a)$).

Proof of the "only if" part. Again the proof is by induction on $|T(S)|$. Trees of cardinality $\aleph_0$ have such F. (It is easy to see, but just the same will be proved later in (3.4).) If $|T(S)|$ is a singular cardinal and all its subtrees of smaller cardinality have such a function, then it follows from the compactness theorem that $T(S)$ has. (The theorem will be proved for this particular case later.) So we assume $|T(S)| = \lambda$ is regular, each subtree of smaller cardinality has such functions but there is no such F for all $T(S)$.

In this case, in each enumeration of the branches of $T(s)$, $\{A_\alpha : \alpha < \lambda\}$, there is a stationary cut $C \subseteq \lambda$ such that $\alpha \in C$ implies $A_\beta \in \operatorname{cl}(T_\alpha)$ for some $\beta \geqq \alpha$ (where $T_\alpha = \{A_\beta : \beta < \alpha\}$ and its closure are the branches having unbounded intersection with its branches). As each branch is a countable subset of λ (w.l.o.g. it is a tree of functions to λ), for each A_α there is $f(\alpha) < \lambda$ bigger than all ordinals in A_α. And there is $D \subseteq \lambda$ closed and unbounded there such that $\alpha \in D$, $\beta < \alpha$ implies $f(\beta) < \alpha$. $E = D \cap C$ is stationary in λ and staisfies: if

$\alpha \in E$, and $\beta \in A_\alpha$ and is not the last element there then $\beta < \alpha$. Looking at the proof of lemma 3.4 in [2] we see that the following claim is actually proved there: *If* there is no ω_1 path in $T(S)$, $T(S) = \{A_\alpha : \alpha < \lambda\}$, λ regular, and there is a stationary $C \subseteq \lambda$ such that if $\alpha \in C$, $\beta \in A_\alpha$ and is not the last there then $\beta < \alpha$, *then* C can be separated into two disjoint stationary sets C_0, C_1 such that $f \in C_0$. $g \in C_1$ implies $f < g$. By Lemma 3.5 in this case $(S, <)$ cannot be represented as a union of $\aleph_0$ well orderings. A contradiction.

THEOREM 2.2. *Let T be a tree, $|T| = \lambda$ a singular cardinal, and the height of T is smaller than λ. If every subtree of T whose cardinality is smaller than λ has a function choosing disjoint final segments for its branches, then there is such a function for T.*

PROOF. One can prove the theorem by checking that if we define freeness for subtrees A, B,: "A/B is free if there is such a function for A whose range does not intersect branches from B", then this freeness satisfies the axioms of the compactness theorem. We give here a straightforward proof.

First we divide the branches of T into $\operatorname{cof}\lambda$ disjoint sets each of cardinality less than λ, $T = \bigcup_{i<\operatorname{cof}\lambda} B_i$, where each B_i generates a subtree of T of cardinality less than λ, so having a function F_{B_i} (choosing disjoint final segments for B_i). We define $\bar{F}$ by $\bar{F}(t) = F_{B_i}(t)$ for the B_i such that $t \in B_i$. If $\bar{F}$ were as desired Range$(\bar{F})$ would be a set of disjoint segments of branches, but there may be intersections between those segments so we have to improve $\bar{F}$. Given a segment in Range$(\bar{F})$ each vertex in it belongs to at most $\operatorname{cof}\lambda$ other branches there. Let μ be a cardinal smaller than λ but not smaller than $\operatorname{cof}\lambda$ and the height of T, and let $X^1_{(t)}$ be the set of all branches in Range$(\bar{F})$ intersecting t, then $|X^1_{(t)}| \leqq \mu$. We define by induction $\langle X^n_{(t)} : n < \omega \rangle$: X_{n+1} is the set of all branches of Range$(\bar{F})$ intersecting a branch in X_n. $X^\omega_{(t)} = \bigcup_n X^{(n)}_t$ is a subtree of Range$(\bar{F})$, its cardinality is less than λ and every branch of $X^\omega_{(t)}$ belongs to $X^\omega_{(t)}$ and Range$(\bar{F})$ intersecting a branch of $X^\omega_{(t)}$ belongs to $X^\omega_{(t)}$, and Range$(\bar{F})$ is divided into disjoint such subtrees, as $s \in X^n_t$, $X^m_s \subseteq X^{m+n}_t$, hence $X_s \subseteq X_1 \cdots$.

Every such subtree $X^\omega_{(t)}$ has a function $F_{(t)}$ as needed and their union defines a "good" function $\bar{\bar{F}}$ for Range$(\bar{F})$. Let $F(t)$ be $\bar{F}(t) \cap \bar{\bar{F}}(t)$ and it will be the desired function for T.

§3. Compactness in regular cardinals

When we try to check compactness for regular cardinals we find that the situation is much more complicated. Most of the results are independent of ZFC and the axioms system for freeness that we used in the singular-cardinals case is no longer sufficient as the behaviour (with respect to compactness in regular cardinals) of algebras satisfying these axioms may change from one algebra to another. In addition to this, not all regular cardinals behave the same when compactness for a given algebra is considered. The model (of set theory, of course) in which compactness is rarest is L—the constructible universe. In L (according to a theorem of Jensen [4]) every regular cardinal λ which is not weakly compact has a subset S, stationary in it, such that for every limit $\delta < \lambda$, $S \cap \delta$ is not stationary in δ. This incompactness property implies incompactness in λ for many other algebras, see [6]. We will show that another property of L, the gap-one two-cardinal theorem for singular cardinals, implies another incompactness in successors of singular cardinals for the algebra we discussed in Theorem 2.2. This implication is due to A. Litman.

On the other hand, assuming the existence of certain large cardinals, given almost any regular cardinal κ, we can construct a model in which κ will be compact for some of the above-mentioned algebras. This is done by collapsing a large cardinal to κ in a way that preserves the compactness property of the large cardinal. (In this way we will get the independence with ZFC + GCH of the gap-one two-cardinal conjuncture for singular cardinals.) This is not surprising, as in L a given infinite algebra has as few as possible subalgebras, so it may happen that none of the small subalgebras reflects the situation in the given algebra, assuming the existence of large cardinals, and collapsing them gives us a model "very rich" with subsets and thus changes the pricture.

I. *Compactness above the continuum*

THEOREM 3.1. *If the existence of a super-compact cardinal is consistent with ZFC, then it is also consistent with ZFC that every algebra satisfying the axioms, which is $2^{\aleph_0}$-free, is free.*

PROOF. A super-compact cardinal is a cardinal κ such that for every $\lambda \leqq \kappa$ there exists a normal κ-complete ultrafilter on $P_\kappa(\lambda)$. It follows that for every model $\langle M, E, R_1, \cdots, R_i \rangle_{i<\alpha<\kappa}$ and every n there is an elementary submodel of it $(M', E, \cdots, R_i, M', \cdots \rangle$ satisfying the same n-order sentences, with parameters, and $a \in |M'|$, $M \models |a| < \kappa$, $b \in a$ implies $b \in |M'|$. Thus every κ-free algebra is free. We will blow $2^{\aleph_0}$ to κ using Levi's forcing, and show that this property of κ remains valid in the new model we get.

CLAIM 3.2. *If $A = \bigcup_{\alpha<\lambda} A_\alpha$, λ is a regular cardinal, $\langle A_\alpha : \alpha < \lambda \rangle$ is increasing and continuous, $|A_\alpha| < \lambda$ for every α, and A is λ free, then A B is not free iff there is a stationary $S \subseteq \lambda$ such that $\alpha \in S$ implies $A_{\alpha+1}/A_\alpha \cup B$ is not free.*

This is a simple consequence of Claim 1.2 and Axiom I.

The model. Let κ be a super-compact in V. P is the set of finite functions from $\kappa \times \omega$ to $\{0,1\}$. P satisfies the C.C.C. thus for G generic in P (over V) every cardinal of V is also a cardinal of $V[g]$, and $V[g] \models 2^{\aleph_0} = \kappa$.

CLAIM 3.3. *In $V[G]$ every algebra (which satisfies axioms) which is $2^{\aleph_0}$-free is free.*

PROOF. Assume this is not true. Let A be an algebra of minimal cardinality satisfying on B: A/B is $2^{\aleph_0}$-free but not free. Let $|A| = \lambda$. We represent A as $\bigcup_{\alpha<\lambda} A_\alpha$, continuous and increasing, and $|A_\alpha| < \lambda$. By the minimality of $|A|$ we get that A/B is $2^{\aleph_0}$-free but not free, so by the compactness theorem for singular cardinals λ is regular. By Claim 3.2 we get a stationary set $S \subseteq \lambda$ such that $\alpha \in S$ implies $A_{\alpha+1}/A_\alpha \cup B$ is not free. By the minimality of $|A|$, $A_{\alpha+1}/A_\alpha \cup B$ is not $2^{\aleph_0}$-free, so let $B_\alpha \subseteq A_{\alpha+1}$ be such that $|B_\alpha| < 2^{\aleph_0}$ and $B_\alpha/A_\alpha \cup B$ is not free. Let us work in V. Let N be the model $\langle H_{(\lambda^+)}$, the set P, $\tilde{S}$, $\langle \tilde{A}_i : i < \lambda \rangle$, $\langle \tilde{B}_\alpha : \alpha < \lambda \rangle$, M and all its functions and relations, and Skolem functions for it$\rangle$ ($\tilde{S}$, $\tilde{A}_i$, $\tilde{B}_\alpha$ are names in V for these elements of $V[G]$). ($H_{(\lambda^+)}$ is the collection of all sets in V of hereditary cardinality less than λ^+.) As κ is super-compact there is a model $M \prec N$ satisfying the same second order sentences that N does, and if $a \in |M|$, $|a| < \kappa$ then $a \subseteq |M|$. Let π be an isomorphism of M to a transitive model $\bar{M}$; for $a \in |M|$ with transitive closer of cardinality less than κ $\pi(a) = a$, so in particular for every $p \in P$ $\pi(p) = p$. Let $\bar{\lambda}$ be $\pi(\lambda)$, "$|M| = \bar{\lambda}$ is a regular cardinal", $\bar{M}$ also satisfies "every set of cardinality less than $\bar{\lambda}$ is an element", so by the transitivity of $\bar{M}$ we get that $\bar{\lambda}$ is a regular cardinal in V. By the C.C.C. of P we can choose for every B_α a maximal set of pairwise incompatible conditions describing it of cardinality $\leqq |B_\alpha| + \aleph_0$ which is less than κ, so w.l.o.g. we can assume each B_α has a name $\tilde{B}_\alpha$ of cardinality less than κ (so $\pi(\tilde{B}_\alpha) = \tilde{B}_\alpha$). We decompose the forcing into two steps: first we use only $P \cap M$ as the set of forcing conditions and $G \cap P \cap M$ as the generic set over it, and then we work in $V[G \cap P \cap M]$ with $P - M$ and $G \cap (P - M)$ as a generic set over it. It is well known that the model we get after these two steps is $V[G]$. As M with $P \cap M$ is an elementary (in second order logic) submodel of $H_{(\lambda^+)}$ with P in $\bar{M}[G \cap P \cap M] = \bar{M}[G]$, $\pi(\tilde{S})$ will become a stationary subset of $\bar{\lambda}$. Let us denote it S^m, $\pi(\tilde{A})$ will become an unfree algebra $A^m = \bigcup_{\alpha<\bar{\lambda}} A^m_\alpha$ and $l \in S^m$ will imply B^m_α / A^m_α is not free. As every set in $\bar{M}(G)$ of cardinality less than $\bar{\lambda}$ has a name of cardinality less than $\bar{\lambda}$, it is true in $\bar{M}(G)$ that every set of cardinality less than $\bar{\lambda}$ is an element, so S^m is a stationary subset of $\bar{\lambda}$ and the algebras are not free in $V[G \cap M \cap P]$ as well. N_0 completes the forcing to get $V[G]$, as $\pi(\tilde{B}_\alpha) = \tilde{B}_\alpha$; it becomes the algebra B_α in $\bar{M}[G]$ and in $V[G]$. $\pi^{-1}(A^{\bar{m}})$ defines an algebra on $\lambda \cap |M|$ (for every function f is the algebra and $\bar{a} \in \lambda |M|$ there are $\tau \in \lambda \cap |M|$ and $p \in P$ forcing $f(\pi^{\bar{m}}(\bar{a})) = (\tau)$, and as $\bar{M}$ is elementary submodel and π isomorphism $p = \pi(p) | f(\bar{a}) = c)$: $\lambda \cap |M| = \pi^{-1}(A^{\bar{m}}) = \bigcup_{i<\bar{\lambda}} \pi^{-1}\pi^{-1}(A_i^{\bar{m}})$ and $S^{\bar{m}}$ is a stationary subset of $\bar{\lambda}$ in $V[G]$ by the C.C.C. for $P \cap |M|$) and for $\alpha \in S^{\bar{m}}$, $\pi^{-1}(B_\alpha)/\pi^{-1}(A_\alpha)$ is not free, so by the first claim $A \cap |M| = \lambda \cap |M|$ is not free, contradicting the $2^{\aleph_0}$ freeness of A (it is an intersection of A with an elementary submodel of $\boldsymbol{M}$ as we had Skolem functions for $\boldsymbol{M}$ our model).

Remark. If fact we proved the theorem for any notion of freeness that can be expressed in an n-order sentence with less than κ parameters, and for which non-freeness is not destroyed by blowing $2^{\aleph_0}$ to κ.

II. Compactness in μ^{++} and the two cardinal conjecture

In this section we show that if the existence of a compact cardinal is consistent with ZFC + GCH then it is also consistent with ZFC + GCH for algebra $A(\mu)$: freeness in μ^{++} implies freeness.

On the other hand, we show that for singular λ *if* every first order sentence which has $\langle \aleph_1, \aleph_0 \rangle$ model has a $\langle \lambda^+, \lambda \rangle$ model *then* there is an algebra $A(\operatorname{cof} \lambda)$ which is λ-free but not free. In particular this shows the independence of the well known conjecture "$\langle \aleph_1, \aleph_0 \rangle \to \langle \lambda^+, \lambda \rangle$ for singular λ" with ZFC + GCH assuming, of course, that "ZFC + GCH + there exists a compact cardinal" is consistent. (Jensen showed that in $V = L$ this two cardinal conjecture holds.)

Definition. An algebra in $A(\mu)$ will be a tree in which every maximal branch is of height μ. For such trees, A, B, A/B will be free if there is a function F assigning to every branch of $A - B$ a final segment of it such that for incomparable branches they are disjoint and no segment in Range(F) intersects a branch of B. (This is the algebra mentioned in §2 above.)

Theorem 3.4 (Ami Litman). *If every first order sentence (with a distinguished predicate p) that has an $\langle \aleph_1, \aleph_0 \rangle$ model has a $\langle \lambda^+, \lambda \rangle$ model and λ is singular, then there is an algebra in $A(\operatorname{cof} \lambda)$ which is λ^+-free but not free.*

Proof. First note that in $A(\aleph_0)$ there is an $\aleph_1$-free non-free algebra. Simply take $\aleph_1$ branches of the tree of increasing sequences of natural numbers (ordered by inclusion). As there are $\aleph_1$ branches and only $\aleph_0$ proper intitial segments of them it is clear we can't choose for every branch a final segment so that these segments will be pairwise disjoint. On the other hand, given any set of $\aleph_0$ branches $\{a_i : i < \omega\}$ we can define a function F as needed by induction of i;

coming to define $F(a_{i+1})$ there are only i branches we must avoid and as a_{i+1} is infinite it can be done.

Let L be a first order language with a distinguished one place predicate symbol P, with $\in$, a two place predicate $<$, and function symbols F_1, F_2, F_3. φ will say:

(1) $\forall_y \forall_z (\forall x(x \in Y \leftrightarrow x \in Z) \rightarrow y = z)$.

(2) $<$ is a linear order of the model.

(3) For every x, $F_1(x)$ is a subset of P $(=\{x : P(x)\})$ cofinal in it.

(4) For every y, $F_2(x, y)$ is a function from $\{x : x < y\}$, $F_2(x, y)$ is a final segment of $F_1(x)$ and for $x, z < y$, $x \neq z$, $F_2(x, y) \cap F_2(z, y) = \emptyset$.

(5) $F_3(y, t)$ is a one one function from P on a cofinal subset of $F_1(y)$ which is order preserving.

It is easy to see that in a tree of $\aleph_1$ increasing sequences of natural numbers F_1, F_2, F_3 can be interpreted so that it will become an $\langle \aleph_1, \aleph_0 \rangle$ model of φ (P will, of course, be the natural numbers).

Let M be a $\langle \lambda^+, \lambda \rangle$ model for φ. First we show that $\operatorname{cof}\{x : P(x)\}$ (in M) is $< \operatorname{cof} \lambda$. We know that $|\{x : P(x)\}| = \lambda$ so let $\{A_i : i < \operatorname{cof} \lambda\}$ be an increasing sequence of subsets of $\{x : P(x)\}$ such that for every i $|A_i| < \lambda$, and $P = \bigcup A_i$. As every $F_1(x)$ is cofinal in P, if $\operatorname{cof} P \neq \operatorname{cof} \lambda$ for every x there is i_x such that $F_1(x) \cap A_{i_x}$ is cofinal in P, so there exists i_0 such that for λ^+ elements x $F_1(x) \cap A_{i_0}$ is cofinal in P. Let y be such that λ of these λ^+ elements are smaller than y. Now for every such x there is an element of $F_1(x) \cap A_{i_0}$ bigger than $F_2(x, y)$ and for different x's these elements are different (by part (4) of φ), so there are λ elements in A_{i_0}, contradiction. Now we take a cofinal sequence P of length $\operatorname{cof} \lambda$, a_α $(\alpha < \operatorname{cf} \lambda)$, and to each x we attach the sequence $\langle F_3(x, a_\alpha)$: $\alpha < \operatorname{cof} \lambda \rangle$.

The tree of all initial segments of those sequences (ordered by inclusion) will give us the desired algebra. F_2 assures the existence of the function F for every set of less than λ^+ branches and F_3 assures that there are only λ proper intial segments of branches, so there cannot be such a function F for all the λ^+ branches together.

THEOREM 3.5. *If "ZFC + (GCH) + there exists a compact cardinal" is consistent, then for every μ so is "ZFC + (GCH) + An algebra in $A(\mu)$ which is μ^{++} free, is free".*

REMARK. We can do it for any μ, and even for many μ's together. In our model we thus get it for $A(\mu')$, $\mu' < \mu$.

PROOF. We start with a model of ZFC in which there is a compact cardinal κ.

By Levi's conditions we collapse it to μ^{++} and then we show, as in [2], that the new model has the desired property.

First note that if κ is a compact cardinal and $\mu < \kappa$ then every κ-free algebra in $A(\mu)$ is free. This follows from the definition of a compact cardinal by the compactness property for $L_{\kappa\kappa}$. (Using a language with individual constants for every branch and vertex in the tree, we can give a complete description of the tree in $L_{\kappa\kappa}$. Let T be the theory describing the tree and saying that F is a function showing its freeness. The freeness of the tree assures every subtheory of cardinality less than κ has a model, so T has a model and we can build F for our tree from the F we get in this model.) So it is enough to deal with μ such that $\mu^{++} < \kappa$.

It is easy to check that our algebras satisfy axioms II–IV and VI, VII so we can use Claim 0.2. We need a little more:

Claim 3.6. *Let λ be a regular cardinal $> \mu$. Let A be a λ-free algebra in $A(\mu)$, $|A| = \lambda$. Given an enumeration of its branches $\langle A_i : i < \lambda \rangle$, A is free iff there is no stationary $C \subseteq \lambda$ such that for $\alpha \in C$ there is an $i \geqq \alpha$ such that A_i is in $\overline{\langle A_j : j < \alpha \rangle}$ (i.e. for every vertex in A_i there is a branch A_β, $\beta < \alpha$, containing it).*

Proof of the Claim. Let T_α be $\{A_i : i < \alpha\}$. If A is free then by Claim 0.2 there is a closed and unbounded $S \subseteq \lambda$ such that for $i_1, j \in S$, $i < j$, T_j / T_i is free. If there is such C then $C \cap S \neq \emptyset$ and for $i \in C \cap S$ there is an $\alpha \geqq i$ such that $A_\alpha \in \bar{T}_i$; taking $j \in S$, $j < \alpha$ we see that T_j / T_i is not free, a contradiction. If A is not free, then there is no closed and unbounded $S \subseteq \lambda$ such that for $\alpha \in S$, A / T_α is λ-free, for if there were such a set we could define a freeness function F for A by stages on the intervals defined by the elements of S (taking unions at limit points), so there is a stationary $C \subseteq \{d : A / T_\alpha$ is not λ-free$\}$. For $\alpha \in C$ there is $\tilde{T} \subseteq A$, $|T| < \lambda$ such that $\tilde{T} / T_\alpha$ is not free, as A is λ-free $\tilde{T}$ is free, and if every branch in $\tilde{T} - \tilde{T}_\alpha$ had a final segment disjoined to all branches of T_α we could build a freeness function for $\tilde{T} / T_\alpha$ by intersecting the freeness function for $\tilde{T}$ with those final segments, so necessarily there is an $A_\beta \in t(\tilde{T} - T_\alpha) \cap \bar{T}_\alpha$, as $A_\beta \not\subseteq T_\alpha$, $\beta \leqq \alpha$ so C is the desired stationary set.

The forcing. Let m be a countable transitive model of "ZFC + there exists a compact cardinal" and let κ be such a cardinal in m.

We define M. Levi's conditions to collapsing κ to μ^{++}. (That is: Let K be the set of regular cardinals greater than μ^+ and smaller than κ. A condition p will be a function from a subset of $\kappa \times \mu^+$ with cardinality smaller than μ^+ to κ such that for every (α, β) in its domain $p(\alpha, \beta) < \alpha$; P is ordered by functions inclusion.)

It is known that P is μ^+-complete, and satisfies the κC.C. From the μ^+-completeness we know that cardinals smaller than μ^{++} in m remain cardinals in $m[G]$. The κC.C. assures cardinals not smaller than κ remain cardinals, and every closed and unbounded set of κ in $m[G]$ contains a closed and unbounded set of κ belonging to m. (In particular stationary sets of κ in m remain stationary in $m[G]$.)

Another important property of this forcing is that for every regular cardinal λ, $\mu^+ < \lambda < \kappa$, if P_λ is the set of conditions whose domain is a subset of $\lambda \times \mu^+$ and p^λ, the set of conditions whose domain is a subset of $(\kappa - \lambda) \times \mu^+$, then $P = P_\lambda \times P^\lambda$; and if G is generic for P over m then $G \cap P_\lambda$ is generic for P_λ over m, $G \cap P^\lambda$ is generic for P^λ over $m[G \cap P_\lambda]$ and $m[G] = m[G \cap P_\lambda][G \cap P^\lambda]$.

THE MAIN CLAIM 3.7. *$m[G] \models$ "every $A \in A(\mu)$ such that $|A| = \mu^{++}$ and A is μ^{++}-free, is free".*

PROOF OF THE CLAIM. W.l.o.g. we can assume A is a tree of increasing sequences of ordinals smaller than κ $(=\mu^{++})$ ordered by inclusion. Let $\langle A_i : i < \mu^{++} \rangle$ be a fixed enumeration of the branches of the tree and let $T: \kappa \times \mu \to \kappa$ the function defining the tree, i.e. $T(\alpha, i)$ is the vertex of A_α. If A is not free we know by Claim 3.2 that there is an $C \subseteq \kappa$ stationary there such that $\alpha \in C \to$ there is an $i \geqq \alpha$ such that $A_i \in \bar{T}_\alpha$; we can enumerate the tree such that it is α.

We work in m. Let $\tilde{C}$ be a name for C and $\tilde{T}$ a name for T. For every pair $\langle \alpha, i \rangle$ let $X_{\alpha i}$ be a maximal set of pairwise incompatible conditions each of them forcing a value for $T(\alpha, i)$. Let X_α be $\bigcup_{i<\mu} X_{\alpha,i}$ by the κC.C. for every α, $|X_\alpha| < \kappa$. Let $f_{(\alpha)}$ be an ordinal such that $p \in X_\alpha$ implies $p \in P_{f(\alpha)}$ and if there is an $i < \mu$ such that $p \,|$"$\tilde{T}(\alpha, i) = \beta$" then $\beta < f(\alpha)$. Let D be the closed and unbounded set in κ such that $\alpha \in D$ and $\beta < \alpha$ implies $f(\beta) < \alpha$.

We consider the model $\langle R_{(\kappa)}, \in, D, \bar{T}, C, P, p_0 \rangle$, where p_0 is "$\tilde{T}$ is a μ^{++}-free, non-free algebra in $A(\mu)$", $\bar{C}$ is a binary relation on $P \times On$, $\langle p, \alpha \rangle \in \bar{C}$ iff $p \Vdash \alpha \in \tilde{C}$, and $\bar{T}$ is a four place relation on $P \times On^3$, $\langle p, \alpha, \beta, \gamma, \rangle \in \bar{T}$ iff $p \Vdash \tilde{T}(\alpha, \beta) = \gamma$.

The following sentence is a π^1_1-sentence satisfied in this model: D is closed and unbounded in κ, for every set E. If E is closed and unbounded in κ then there exists q extending p_0 and an ordinal α such that $q \Vdash$ "$\alpha \in \tilde{C} \cap E$". As κ is compact, it is π^1_1-indescribable so there exists a strongly inaccessible $\lambda < \kappa$ such that $\langle P_{(\lambda)}, E, D \cap \lambda, \bar{C} \cap (p \times \lambda), \bar{T} \cap (p \cap \lambda \times \mu \times \lambda) \rangle$ is an elementary submodel of the above model and satisfies this sentence.

CLAIM 3.8. *$T_\lambda = \{A_\alpha : \alpha < \lambda\}$ belongs to $m[G \cap P_\lambda]$ and is not free there.*

PROOF. As $D \cap \lambda$ is unbounded in λ and D is closed, $\lambda \in D$ so for every $\alpha \in R_{(\lambda)}$, $X_\alpha \subseteq P_\lambda$, as G is generic for every $\alpha < \lambda$, $i < \mu$ there exists a unique $q \in X_{\alpha i} \cap G$ forcing a value for $\tilde{T}(\alpha, i)$, $q \in G \cap P_\lambda$ so every A_α, $\alpha < \lambda$ is definable in $m[G \cap P_\lambda]$, $T_\lambda \in m[G \cap P_\lambda]$. $C \cap \lambda = \{\alpha < \lambda : \exists q \in G \cap P_\lambda$, $q \Vdash \alpha \in \tilde{C}\}$ so it also belongs to $m[G \cap P_\lambda]$ and it is stationary there (by the π^1_1-sentence, and the fact that sets stationary in λ in m remain such sets in $m[G \cap P_\lambda]$). $\alpha \in C \Rightarrow A_\alpha \in \bar{T}_\alpha$ so T_λ is not free in $m[G \cap P_\lambda]$.

CLAIM 3.9. *T_λ is not free in $m[G]$.*

In order to prove this, note that in $m[G \cap P_\lambda]$ $\lambda = \mu^{++}$, and as P_λ is μ^{++}-complete, no sets of cardinality smaller than μ^+ are in $m[G \cap P_\lambda]$-m so if we define Levi's conditions to collapse κ to μ^{++} in $m[G \cap P_\lambda]$ we get $P \cap ((\kappa - \lambda) \times \mu^+)$.

So it is enough to prove

LEMMA 3.10. *Let m be a model in which λ is a regular cardinal, $T = \{A_\alpha : \alpha < \lambda\}$ an algebra in $A(\mu)$, $\mu < \lambda$, and $C \subseteq \lambda$ is stationary there such that $\alpha \in C \Rightarrow A_\alpha \in \bar{T}_\alpha$, then forcing above m by a μ^+-complete set of conditions will not make T free.*

PROOF OF THE LEMMA. Let p_0 force that $\tilde{F}$ is a freeness function of T. We attach a condition p_η to every vertex $\eta \in T$ such that $p_\eta > p_0$, and $\eta' > \eta$ (in T) implies $p_{\eta'} \geqq p_\eta$, and $p_\eta \Vdash$ "η is not the first element in any $\tilde{F}(\eta')$" or $p_\eta \Vdash$ "η is the first element in $F(A_{f(\eta)})$" where $f(\eta)$ is an ordinal smaller than λ. This is possible as the height of every branch is μ and the union of an increasing chain of μ-conditions is still a condition, so in every step we have to extend one condition only. As $\operatorname{cof} \lambda > \mu$ for every branch B the set of $f(\eta)$ for $\eta \in D$ is bounded, so there exists a closed and unbounded set $D \subseteq \lambda$ such that $\alpha \in D \Rightarrow$ if $\eta \in A_\beta$ for $\beta < \alpha$ then $f(\eta) < \alpha$. Let $\eta \in D \cap C$, $p_\xi = \bigcup_{\eta \in A_\xi} p_\eta$. p_ξ is a condition extending p_0, but p_0 forced F is a freeness function, where p_ξ forces $F(A_\xi)$ cannot be defined as needed. So there is no p_0 forcing T is free, and T is not free in $m[G]$.

CONCLUSION. In $m[G]$ A is not μ^{++}-free because in $m[G]$ $T_\lambda \subseteq A \mid T_\lambda \mid < \mu^{++}$ and T_λ is not free. So we got a contradiction and finish the proof of Main Claim 3.7.

THE FINAL CLAIM 3.11. $m[G] \models$ "*every $A \in A(\mu)$ which is μ^{++}-free is free*".

PROOF. Let A be such an algebra. We decompose it to κ trees each of which belongs to m, as follows: To every $p \in G$ let T_p be the subalgebra of that p

which already forces what it will be. First note that for every p, T_p is free in $m[G]$ because: If $|T_p| < \mu^{++}$ it is free by the μ^{++}-freeness of p, otherwise $|T_p| \leqq \kappa$; by the compactness of κ if T_p is not free in m it has a subtree of cardinality smaller than κ which is not free in m. Such a subtree with a minimal cardinality will still be non-free in $m[G]$ (by the lemma) contradicting the μ^{++}-freeness of A. So T_p is free in m and of course in $m[G]$.

Now we build a freeness function for A. For every T_p we choose a freeness function for it, F_p. We define a function F' by well ordering the conditions and for every A branch B is p is the first condition in this order such that $B \in F_p$, $F' = F_p(B)$. Of course F' must not be a good function but as there are only κ trees T_p and w.l.o.g. $\mathrm{cof}(A) > \kappa$ we finish as in Theorem 2.2.

A similar theorem for stationary sets

THEOREM 3.12. *If "ZFC + there exists a weakly compact cardinal (super compact)" is consistent, then so is "ZFC + for every stationary set C in* μ^{++} *whose elements have cofinality* $\leqq \mu$ *(for every stationary set C whose elements have cofinality* $\leqq \mu$*) there is a limit* $\delta < \mu^{++}$ *such that* $C \cap \delta$ *is stationary in* δ*".*

PROOF. The proof is very much like that of Theorem 3.5. We build the same model $m[G]$. We can also repeat the first part of that proof (building $m[G \cap P_\lambda]$) as all we used to get it was the weakly compactness of κ. If we assume κ is super compact we can get such a model also for a stationary set in a cardinal bigger than κ.

To complete the proof, we have to show that by forcing with a set of conditions which is μ^+-complete the set C remains stationary in λ. (This is the analogue of Claim 3.9.) To show this we build a tree of increasing sequences of ordinals such that:

(1) For every $\alpha \in C$ there is a branch A_α whose ordinals converge to α and for every proper initial segment of it there is another branch in the tree containing it and converging to some $\beta < \alpha$.

(2) For every $\alpha < \lambda$ there are less than λ branches converging to it in the tree.

(3) The height of every branch is not more than μ.

(If for every $\alpha < \lambda$, $\alpha^\mu < \lambda$ then all the increasing μ-sequences in λ form such a tree.)

Let p_0 be a condition forcing $\tilde{S}$ is a name of a closed and unbounded subset of C and $S \cap C = \varnothing$. For every vertex η in the tree, we attach a condition p_η extending p_0 and the conditions attached to vertices below it, such that if β is the ordinal in η, p_η forces that an ordinal $f(\beta) > \beta$ will be the first element of S

above β_1 by (2), and by the regularity of λ there is a closed and unbounded $D \subseteq \lambda$ such that for $\alpha \in D$ and β in a branch converging to a limit smaller than α, $f(\beta) < \alpha$. For $\alpha \in D \cap C$, taking A_α as described in (1), $p_\alpha = \bigcup_{\eta \in A_\alpha} p_\eta$ is a condition extending p_0 but forcing $\alpha \in S$ because for every β in A_α, $f(\beta) < \alpha$, so S is unbounded bleow α, and p_0 forces it to be closed, a contradiction.

§4. Related answers and problems

4.1. In [2], problem 2 asks if it is provable in ZFC that there exists a set $S \subseteq N$ such that $|S| = W_{\omega+1}$, $s(f) = w_\omega$ for all $f \in S$, and for all $S' \subseteq S$ if S' is uncountable then so is $\{g \in N$: for some $f \in S'$, $g \subseteq f\}$.

Let us remind the reader that N is the class of strictly increasing continuous functions from a successor countable ordinal to the ordinals, and for $f \in S$, $s(f)$ is sup(Range f).

We have a partial answer: In $V = L$ such a set S exists. This is an easy consequence of Theorem 3.4.

4.2. In [1] a logic $L_{(a,a)}$ is introduced. This is the logic obtained from the usual first order logic by allowing a second order quantifier "aaS" which means for a closed and unbounded set of countable sets s.

Problem 9.3 asks: Does every (standard) model for $L(aa)$ have an elementary (for this logic) submodel of cardinality $\aleph_1$? We show that the answer to this problem is independent of the axioms ZFC.

(A) Assuming $V = L$ there is a model for $L(aa)$ that has no elementary submodel of cardinality $\aleph_1$.

The model will be $\langle \omega_2, <, p \rangle$, where $<$ is the order of the ordinals and p is a unary predicate such that $\{x : p_{(x)}\}$ is stationary in ω_2; $p_{(x)}$ implies $\mathrm{cof}(x) = \omega$ and for every $\alpha < \omega_2$ $\{x : p(x) \wedge x < \alpha\}$ is not stationary in α. By Jensen [4] such a set exists. Now, in this model the set of all countable subsets whose supremum belongs to $\{x : p_{(x)}\}$ is stationary (in the sense of $L(aa)$); this is a sentence of $L(aa)$ which is not true in any elementary submodel of cardinality $\aleph_1$.

(B) Assuming the existence of a supercompact cardinal, we get, just like in Theorem 3.12, a model of ZFC in which every model for $L(aa)$ does have an elementary submodel of cardinality $\leqq \aleph_1$.

4.3. In [6] Shelah shows that assuming $V = L$ for every regular $\mu < \lambda$, λ is not weakly compact. There is a graph G of cardinality λ such that every subgraph of it of smaller cardinality has a colouring number $\leqq \mu$ but G does not.

We show this incompactness result applies also to directing numbers of graphs by showing:

THEOREM 4.3. *Let G be a graph and η a cardinal. Then G has colouring number $\leqq \eta$ iff it has a directing number $\leqq \eta$.*

PROOF. Recall that G has colouring number $\leqq \eta$ if there is a well ordering $<$ of its set of vertices such that $\{|b < a : b \in G(a, b)$ is an edge of $G\}| < \eta$ for each $a \in G$.

G has a directing number $\leqq \eta$ if its edges can be directed such that the number of edges going *from* each vertex is smaller than μ.

It is clear that if G has colouring number $\leqq \eta$ so is also its directing number; one simply sets every edge to go from the high vertex to the small one.

The other direction is proved by induction on $|G|$. If $|G| \leqq \eta$ there is nothing to prove. If $|G| = \lambda > \eta$ we can represent G as $\bigcup_{i<\lambda} G_i$ such that for each i, $|G_i| < \lambda$, $\langle G_i : i < \lambda \rangle$ is increasing, and if (after G is directed as assumed it can be) from $a \in G_i$ there is an edge going to b, b also belongs to G_i. Now we well order $G_i - \bigcup_{j<i} G_j$ for each $i < \lambda$ such that its colouring number is $\leqq \eta$ (using the induction hypothesis) and for $a \in G_i$, $b \notin G_i$ we define $a < b$.

4.4. QUESTION. Is it consistent with ZFC that for every stationary subset S of $\aleph_{\omega+1}$ there is a limit $\alpha < \aleph_{\omega+1}$ such that $S \cap \alpha$ is stationary in α? Note that by Theorem 3.12 it is true if $\{\operatorname{cof} \alpha : \alpha \in S\}$ is bounded below $\aleph_\omega$ and by [4] it is not consistent with ZFC + $V = L$.

4.5. QUESTION. Is it consistent with ZFC that there is a group of cardinality $\aleph_{\omega+1}$ which is not free but every subgroup of it with smaller cardinality is free?

REMARK. It seems that the two questions are closely related one to the other.

ACKNOWLEDGMENT

I wish to thank Prof. Baumgartner for calling my attention to a mistake in §2, in an earlier version of this paper.

Added in proof. For the cases where $Ax\, I^*$ holds, Shelah has found a much shorter proof for the compactness theorem; for the singular case see [9].

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